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Classical Particle Dynamics in Quantum Space

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It is suggested that if space-time is quantized at small distances, then even at the classical level particle motion in space is complicated and described by a nonlinear equation. In the quantum space the Lagrangian function or energy of the particle consists of two parts: the usual kinetic terms, and a rotation term determined by the square of the inner angular momentum—a torsion torque caused by the quantum nature of space. Rotational energy and rotational motion of the particle disappear in the limit $l \to 0$, where l is the value of the fundamental length. In the free particle case, in addition to the rectilinear motion, the particle undergoes a rotation given by the inner angular momentum. Different possible types of particle motion are discussed. Thus, the scheme may shed light on the appearance of rotating or twisting, stochastic, and turbulent types of motion in classical physics and, perhaps, on the notion of spin in quantum physics within the framework of the quantum character of space-time at small distances.

1. INTRODUCTION

In previous papers (Namsrai, 1985b; Dineykhan and Namsrai, 1985) we have introduced quantum space-time and considered some of its interesting consequences. Our method of introducing quantum space-time may be regarded as a local general coordinate transformation:

$$x^{\mu} \Rightarrow \hat{x}^{\mu} = x^{\mu} + l\Gamma^{\mu}(x) \tag{1}$$

where $\Gamma^{\mu}(x)$ are arbitrary noncommutative functions of the points x^{μ}, and l represents the value of the fundamental length.

The attraction of the approach based on the hypothesis of quantum space-time (1) is that it gives rise to the appearance of space-time torsion and to the existence of magnetic monopoles. The latter two facts may be understood as follows; whole space-time on a large scale continued from

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the microscale and obtained by an averaging procedure in which its quantum character appears differs from the usual space-time structure, where there is no place for magnetic monopoles described by some regular potential $\mathbf{A}(x)$, since divrot $\mathbf{A} \equiv 0$ in it. In contrast, as we have shown (Dineykhan and Namsrai, 1985), in space-time with torsion, divrot $\mathbf{A} \neq 0$ for a regular magnetic monopole potential, say $\mathbf{A} = (\boldsymbol{\sigma}/i)g(1/r)$, where g is the value of magnetic charge.

We assume that this structural difference of whole space-time on a large scale continued from the microscale must have an effect on particle behavior, even at the classical level. In this paper, we consider this problem and study particle dynamics for the nonrelativistic case. It turns out that in our scheme the Lagrangian function of a classical free particle is determined by the usual kinetic part and an additional term connected with the rotational degree of freedom corresponding to an inner angular momentum (or sector velocity $\mathbf{v}_s = \frac{1}{2}[\mathbf{r}; \mathbf{v}]$) caused by the quantum structure of space. Due to the latter term, the dynamics of the particle is changed and is determined by a nonlinear differential equation. In the simple case of two-dimensional space and free motion, the derived equation of motion is integrated completely. The initial value problem of this equation is investigated. Depending on initial conditions, a particle's trajectory is complicated and the particle makes a spiral-like motion along the direction of the classical rectilinear trajectory.

However, it is generally difficult to solve an equation of motion in quantum space by analytic methods, and numerical integration is needed. The resulting particle trajectory is very tortuous and, it seems to behave like a strange attractor, at least in the domain determined by the parameter l. We know that the strange attractor is the direct image of "finite" turbulence characterized by a continuous spectrum over time and it is also the mathematical image of stochastic auto-oscillation. So, it is hoped that our approach may be useful to understand the origin of twisting, stochastic, and turbulent processes in physics. However, in order to shed light on this problem, further careful study is needed.

In Section 2 we obtain the Lagrangian function for free particles in quantum space and the Euler-Lagrange equation by using the action principle for the particle trajectory in large-scale, nonquantum space. The concrete form of the motion equation is obtained in Section 3. Sections 4 and 5 are devoted to the study of the Cauchy problem for the obtained equation of motion in two-dimensional space. Here integration of the motion equation is carried out explicitly and some interesting possible types of particle motion due to quantum space are also considered. In Section 6 we discuss the results in order to generalize the given scheme to the relativistic and quantum mechanical cases.

2. THE LAGRANGIAN FUNCTION AND THE ACTION PRINCIPLE FOR A NONRELATIVISTIC PARTICLE IN QUANTUM SPACE

In the nonrelativistic case, we suggest that the quantum character of space-time is manifested only through spatial variables; i.e., the coordinates of quantum space at small distances consist of two parts [like (1)]

$$x^i \Rightarrow \hat{x}^i = x^i + l\Gamma^i(x), \qquad i = 1, 2, 3 \tag{2}$$

and time is an ordinary continuous c-number variable here.

Further, we assume that all physical quantities characterizing a particle's state depend on quantum variables $\hat{x}^i$, $\dot{\hat{x}}^i$, and t; in particular, the Lagrangian function of the free particle is constructed by the corresponding principle as in classical mechanics,

$$\mathscr{L}(\hat{x}^i, \dot{\hat{x}}^i) \equiv \mathscr{L}(\dot{\hat{x}}^i) = m(\dot{\hat{x}}^i)^2/2 \tag{3}$$

where $\dot{\hat{x}}^i = d\hat{x}^i/dt$ is a velocity-like vector in quantum space, and m is the mass of the particle. It should be noted that real, observable particle motion over time t takes place in the nonquantum space x^i on a large scale continued from the microscale where its quantum property is manifested. Thus, in order to go over to a large scale we must carry out some averaging procedure over the microscale [for details, see Namsrai (1985a,b)]. In the concrete case where the functions $\Gamma^i(x)$ in (2) are given in matrix form, the averaging procedure is reduced to taking traces of matrices; for example, if $\Gamma^i(x) \sim \sigma^i$ (σ^i are the Pauli matrices)

$$\langle \hat{x}^i \rangle^{Df} \equiv (1/d)\,\mathrm{Tr}(\hat{x}^i) = x^i \tag{4}$$

where the parameter d arises from the normalization condition; in given case, $d = 2$, since σ^i are two column matrices.

Now choose the matrix form

$$\Gamma^i(x) = \sigma^a e^i_a(x) \tag{5}$$

in (2) [where $e^i_a(x)$ are the tetrad fields (a, $i = 1, 2, 3$)] and study expression (3) in the whole space at large distances. For this, first we define the generalized velocity of the particle by the formula

$$\frac{d\hat{x}^i}{dt} = \frac{dx^i}{dt} + l\sigma^a \frac{de^i_a(x)}{dx^j}\frac{dx^j}{dt} \tag{6}$$

in accordance with (2) and (5). To calculate the Lagrangian (3) over the large scale, expression (6) should be squared and averaged. As a result, we have

$$\mathscr{L}(x^i, \dot{x}^i) = \langle \mathscr{L}(\hat{x}^i, \dot{\hat{x}}^i)\rangle = \frac{1}{d}\operatorname{Tr}(\mathscr{L}(\hat{x}^i, \dot{\hat{x}}^i) = \frac{1}{2}\operatorname{Tr}(\mathscr{L}(\hat{x}^i, \dot{\hat{x}}^i))$$

$$= \frac{m(\dot{x}^i)^2}{2} + \frac{ml^2}{2}\frac{de^i_a}{dx^j}\frac{de^i_a}{dx^k}\dot{x}^j\dot{x}^k \tag{7}$$

As in the case of classical mechanics, with the Lagrangian function (7) we can formulate the law of motion of mechanical systems by using the action principle (or Hamilton's principle) (see, for example, Landau and Lifshitz, 1965).

At times $t = t_1$ and $t = t_2$ let the mechanical system (in the given case, a mechanical material point) occupy definite positions characterized by two sets of coordinate values $x^{(1)}$ and $x^{(2)}$. Then between them the system moves so that the action integral

$$S = \int_{t_1}^{t_2} dt\, \mathscr{L}(x^i, \dot{x}^i)$$

takes the smallest possible quantity, where $\mathscr{L}(x^i, \dot{x}^i)$ is given by formula (7). As in the usual classical mechanical case, for our scheme the action principle tells us that

$$\delta S = \delta \int_{t_1}^{t_2} dt\, \mathscr{L}(x^i, \dot{x}^i) = 0$$

and carrying out the variation, we obtain the Euler–Lagrange equation of a nonrelativistic particle,

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{x}^i} - \frac{\partial \mathscr{L}}{\partial x^i} = 0, \qquad i = 1, 2, 3 \tag{8}$$

3. EQUATION OF MOTION FOR A FREE PARTICLE IN QUANTUM SPACE

Now our aim is to obtain an equation of motion of a free nonrelativistic particle from the Euler–Lagrange equation (8) with the function (7). For this, a concrete form of the tetrad field $e^i_a(x)$ should be defined. As in a previous case (Dineykhan and Namsrai, 1985), we choose a spherical frame of reference as the tetrad coordinate system and a Cartesian frame of reference for the world coordinate system. Then, in the nonrelativistic case the tetrad field $e^i_a = e^i_a(x(t))$ has the form

$$e^i_a = \partial \xi^i / \partial x^a, \qquad e^a_i = \partial x^a / \partial \xi^i$$

where

$$d\xi^1 = dr, \qquad d\xi^2 = r\,d\theta, \qquad d\xi^3 = \rho\,d\varphi$$
$$dx^1 = dx, \qquad dx^2 = dy, \qquad dx^3 = dz$$

and $r = (x^2 + y^2 + z^2)^{1/2}$, $\rho = (x^2 + y^2)^{1/2}$. One can easily see that the field e_i^a is given by

$$e_i^a = \begin{pmatrix} x/r & y/r & z/r \\ zx/r\rho & zy/r\rho & -\rho/r \\ -y/\rho & x/\rho & 0 \end{pmatrix} \tag{9}$$

In this case, the square of the generalized velocity

$$\langle(\dot{\hat{x}}^i)^2\rangle = \langle\dot{\hat{x}}^2\rangle + \langle\dot{\hat{y}}^2\rangle + \langle\dot{\hat{z}}\rangle$$

takes the form

$$\langle\dot{\hat{x}}^2\rangle = \dot{x}^2 + \frac{l^2}{r^4}[(\dot{x}y - y\dot{x})^2 + (y\dot{z} - z\dot{y})^2 + (z\dot{x} - x\dot{z})^2]$$

$$\langle\dot{\hat{y}}^2\rangle = \dot{y}^2 + \frac{l^2}{r^4}\left[(z\dot{x} - x\dot{z})^2 + (y\dot{z} - z\dot{y})^2 + \frac{z^4}{\rho^4}(x\dot{y} - y\dot{x})^2\right]$$

$$\langle\dot{\hat{z}}^2\rangle = \dot{z}^2 + \frac{l^2}{\rho^4}(x\dot{y} - y\dot{x})^2$$

Therefore the averaged Lagrangian function (7) acquires the following form:

$$\mathscr{L} = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{ml^2}{r^4}[(x\dot{y} - y\dot{x})^2 + (y\dot{z} - z\dot{y})^2$$
$$+ (z\dot{x} - x\dot{z})^2] + \frac{ml^2}{\rho^4}\frac{z^2}{r^2}(x\dot{y} - y\dot{x})^2 \tag{10}$$

The last two terms may be rewritten in the form

$$R_l = \frac{l^2}{mr^4}\mathbf{M}^2 + \frac{l^2}{m\rho^4}\left(\frac{z}{r}\right)^2 M_z^2$$

where

$$\mathbf{M}^2 = M_x^2 + M_y^2 + M_z^2, \qquad M_z = xp_y - yp_x$$
$$M_y = zp_x - xp_z, \qquad M_x = yp_z - zp_y, \qquad \mathbf{p} = m\mathbf{v}$$

Thus,

$$\mathscr{L} = T + R_l = \frac{m\mathbf{v}^2}{2} + \frac{l^2\mathbf{M}^2}{r^4 m} + \frac{l^2 z^2 M_z^2}{\rho^4 r^2 m} \tag{11}$$

where the vectors **v** and **M** (M_z) are particle velocity and angular momentum (its third component). We see that in our case the Lagrangian function of the free particle takes two parts: the usual kinetic one T and an additional rotation term R_l due to the quantum nature of space at small distances. On the other hand, the function (11) does not depend on time explicitly and therefore the energy of the particle

$$E = \dot{x}^i \frac{\partial \mathscr{L}}{\partial \dot{x}^i} - \mathscr{L} = T + R_l$$

is conserved.

It is clear that because of the last two terms in (11), the equation of motion for a free particle is complicated in our case and takes the form

$$m\ddot{x}^i + ml^2 \left[\frac{\partial^2 e_a^k}{\partial x^j \, \partial x^m} \dot{x}^j \dot{x}^n + \frac{\partial e_a^k}{\partial x^j} \ddot{x}^j \right] \frac{\partial e_a^k}{\partial x^i} \tag{12}$$

where the tetrad field $e_a^k(x)$ is given by (9) ($i, n, m, k, a = 1, 2, 3$). This equation of motion is obtained from the Euler–Lagrange equation (8) and we write it in components:

$$\begin{aligned}
m\ddot{x} &+ \frac{2ml^2}{r^4} \left[y(\ddot{x}y - \ddot{y}x) \left(1 + \frac{r^2 z^2}{\rho^4} \right) + z(\ddot{x}z - \ddot{z}x) \right] \\
&+ \frac{2ml^2}{r^4} (\dot{x}y - \dot{y}x) \left\{ [x(\dot{x}y - \dot{y}x) + 2(z\dot{z} - \mathbf{r}\dot{\mathbf{r}})y] \left(\frac{z^2}{\rho^4} + \frac{2z^2 r^2}{\rho^6} \right) \right. \\
&\left. \times \frac{2z(y\dot{z} + \dot{y}zr^2/\rho^2)}{\rho^2} \right\} \\
&+ \frac{4ml^2}{r^6} \{ z\dot{z}[x(\mathbf{r} \cdot \dot{\mathbf{r}}) - \dot{x}r^2] + (x\dot{x} + y\dot{y})[z(x\dot{z} - z\dot{x}) - y(\dot{x}y - \dot{y}x)] \} = 0 \\
m\ddot{y} &- \frac{2ml^2}{r^4} \left[x(\ddot{x}y - \ddot{y}x) \left(1 + \frac{r^2 z^2}{\rho^4} \right) - z(\ddot{y}z - \ddot{z}y^2) \right] \\
&+ \frac{2ml^2}{r^4} (\dot{x}y - \dot{y}x) \left\{ [y(\dot{x}y - \dot{y}x) + 2x(z\dot{z} - \mathbf{r} \cdot \dot{\mathbf{r}})] \left(\frac{z^2}{\rho^4} + \frac{2z^2 r^2}{\rho^6} \right) \right. \\
&\left. - \frac{2z}{\rho^2} \left(x\dot{z} + \frac{\dot{x}zr^2}{\rho^2} \right) \right\} + \frac{4ml^2}{r^6} \{ z\dot{z}[y\mathbf{r} \cdot \dot{\mathbf{r}} - \dot{y}r^2] \\
&+ (x\dot{x} + y\dot{y})[z(y\dot{z} - z\dot{y}) - x(\dot{y}x - \dot{x}y)\} = 0 \\
m\ddot{z} &- \frac{2ml^2}{r^4} [x(\ddot{x}z - \ddot{z}x) + y(\ddot{y}z - \ddot{z}y)] - \frac{2ml^2}{r^4} \frac{z}{\rho^2} (\dot{x}y - \dot{y}x)^2 \\
&+ \frac{4ml^2}{r^6} \{ (x\dot{x} + y\dot{y})[z\mathbf{r} \cdot \dot{\mathbf{r}} - \dot{z}r^2] - z\dot{z}^2 \rho^2 \} = 0
\end{aligned} \tag{13}$$

We see that these equations of motion are too complex to be solved by analytic methods and require numerical investigation. However, there is a concrete case for which the equation of motion in quantum space can be integrated completely. That is the situation when the particle motion along one of the directions of the coordinate system is rectilinear and becomes complicated along the other two directions due to the quantum structure of space, i.e., it is equivalent to the two-dimensional case. To prove this, we choose the cylindrical frame of reference as the tetrad coordinate system. In this case, instead of (9), we have

$$e_i^a = \begin{pmatrix} x/\rho & y/\rho & 0 \\ -y/\rho & x/\rho & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

By using this tetrad field, we can easily calculate the Lagrangian (11) and equation of motion (13):

$$\mathcal{L} = m\mathrm{v}^2/2 + l^2 M_z^2/\rho^4 m$$

and

$$m\ddot{z} = 0$$

$$m\ddot{x} + 2ml^2 y(\ddot{x}y - \ddot{y}x)/\rho^4 - 4ml^2 y(\dot{x}y - \dot{y}x)(x\dot{x} + y\dot{y})/\rho^6 = 0 \tag{14}$$

$$m\ddot{y} - 2ml^2 x(\ddot{x}y - \ddot{y}x)/\rho^4 + 4ml^2 x(x\dot{y} - \dot{y}x)(x\dot{x} + y\dot{y})/\rho^6 = 0$$

From these equations, we see that the particle moves in a rectilinear way along the z axis, while its motion along x and y axes is complicated and twisted in accordance with (14). We now study a particle trajectory determined by equations (14).

4. INTEGRATION OF THE EQUATION OF MOTION IN THE TWO-DIMENSIONAL CASE

It is convenient to study the equations of motion (14) in polar coordinates (φ, ρ), in which the Lagrangian function has the simple form

$$\mathcal{L} = \tfrac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2) + ml^2\dot{\varphi}^2 \tag{15}$$

This function does not contain the coordinate φ explicitly. Any generalized coordinate q_i not entering explicitly into the Lagrangian function is called cyclic. Due to the Euler–Lagrange equation for such coordinates, we have

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

i.e., the corresponding generalized momentum $p_i = \partial\mathscr{L}/\partial\dot{q}_i$ is the integral of motion. This situation leads to an essential simplification of the integration problem of the equation of motion in the presence of cyclic coordinates.

In the given case, the generalized momentum is

$$p_\varphi = (mr^2 + 2ml^2)\dot{\varphi}$$

The first term coincides with the angular momentum $M_z = m(x\dot{y} - y\dot{x}) = mr^2\dot{\varphi}$. Thus, in our scheme the general angular momentum of the type

$$M = M_z + M_l = (mr^2 + 2ml^2)\dot{\varphi} = \text{const} \tag{16}$$

is conserved.

The equation of motion obtained by using the Lagrangian function (15) takes the following form;

$$\begin{aligned} m\ddot{z} &= 0 \\ m\ddot{\rho} - m\dot{\varphi}^2\rho &= 0 \\ m(\rho^2 + 2l^2)\ddot{\varphi} + 2m\rho\dot{\rho}\dot{\varphi} &= 0 \end{aligned} \tag{17}$$

From the second equation in (17) we have

$$\frac{\ddot{\varphi}}{\dot{\varphi}} = -\frac{2\rho\dot{\rho}}{\rho^2 + 2l^2}$$

or

$$\frac{\partial}{\partial t}(\ln\dot{\varphi}) = -\frac{\partial}{\partial t}[\ln(\rho^2 + 2l^2)]$$

Direct integration of the last equation gives

$$\dot{\varphi} = C_1/(\rho^2 + 2l^2) \tag{18}$$

where the integration constant C_1 is determined by the initial conditions

$$\rho(t)|_{t=0} = \rho_0 \equiv al, \qquad \varphi(t)|_{t=0} = 0, \qquad \dot{\varphi}|_{t=0} = \omega_0 \tag{19}$$

From which $C_1 = (a^2 + 2)\omega_0 l^2$ and expression (18) acquires the form

$$\dot{\varphi} = (a^2 + 2)\omega_0 l^2/(\rho^2 + 2l^2) \tag{20}$$

It should be noted that in the usual case, when $l = 0$, we obtain the rectilinear trajectory given by a ray $\varphi = \varphi_0 = \text{const}$ and $\rho(t) = \rho_0 + v_0 t$ along which classical particles move. Here the parameters ρ_0 and v_0 are given by the initial conditions (19) and

$$\left.\frac{\partial\rho}{\partial t}\right|_{t=0} = v_0 \tag{21}$$

Further, substituting expression (20) into the first equation in (17), we get

$$\ddot{\rho}-\rho(a^2+2)^2\omega_0^2l^4/(\rho^2+2l^2)^2=0$$

To integrate this equation, put $p(\rho)=\dot{\rho}$ and write the new equation for $p(\rho)$

$$p(\rho)\,\partial p(s)/\partial\rho=\rho(a^2+2)^2\omega_0^2l^4/(\rho^2+2l^2)^2$$

Simple integration of this gives

$$\dot{\rho}^2=-(a^2+2)^2\omega_0^2l^4/(\rho^2+2l^2)+C_2 \tag{22}$$

where the integration constant

$$C_2=v_0^2/2+(a^2+2)\omega_0^2l^2/2 \tag{23}$$

arises from the initial conditions (19) and (21). After separating integration variables, equation (22) with (23) may be rewritten in the form

$$\pm\frac{d\rho}{[v_0^2+(a^2+2)\omega_0^2l^2-(a^2+2)^2\omega_0^2l^4/(\rho^2+2l^2)]^{1/2}}=dt \tag{24}$$

We notice that the plus and minus signs in (24) are not important and we choose a plus sign and integrate this equation. For this, we put

$$\rho=2^{1/2}l\operatorname{ctg}x,\quad d\rho=-2^{1/2}l\sin^{-2}x\,dx,\quad x=\operatorname{arcctg}(\rho/l2^{1/2})$$

As a result of this change of variables and integration, we have

$$-\frac{2^{1/2}l}{[v_0^2+(a^2+2)\omega_0^2l^2]^{1/2}}\int dx\frac{\sin^{-2}x}{(1-k^2\sin^2x)^{1/2}}=t+C_3 \tag{25}$$

where

$$k=(a^2+2)\omega_0l2^{-1/2}[v_0^2+(a^2+2)\omega_0^2l^2]^{-1/2} \tag{26}$$

It is easily verified that the integral (25) is reduced to normal elliptic integrals of the first kind

$$F(\varphi,k)=\int_0^{\varphi}dx\,(1-k^2\sin^2x)^{-1/2}$$

and second kind

$$E(\varphi,k)=\int_0^{\varphi}dx\,(1-k^2\sin^2x)^{1/2}$$

Thus, the integral (25) results in

$$\frac{2^{1/2}l}{[v_0^2+(a_0^2+2)\omega_0^2l^2]^{1/2}}\left[\frac{\rho}{2^{1/2}l}\left(1-\frac{k^22l^2}{\rho^2+2l^2}\right)^{1/2}+E\left(\operatorname{arcctg}\left(\frac{\rho2^{-1/2}}{l},k\right)-F\left(\operatorname{arcctg}\frac{\rho2^{-1/2}}{l},k\right)\right]=t+C_3 \tag{27}$$

where the integration constant may be calculated by using (19),

$$C_3 = \frac{2^{1/2} l}{v_0^2 + (a^2+2)\omega_0^2 l} \{a 2^{1/2}[1 - 2k^2(a^2+2)^{-1/2}] + E(\operatorname{arcctg}(a 2^{-1/2}), k) - F(\operatorname{arcctg}(a 2^{-1/2}), k)\}$$

Furthermore, rewriting (20) in the form

$$d\varphi = \omega_0(\rho_0^2 + 2l^2)\, dt/(\rho^2 + 2l^2)$$

substituting dt from (24) with a plus sign, and integrating, we get

$$\varphi = 2^{1/2} lk \int d\rho\, (\rho^2 + 2l^2)^{-1/2}(\rho^2 + 2l^2 - 2l^2k^2)^{-1/2} + \varphi_0$$

where the parameter k is given by (26). Making use of the change of the integration variable

$$\rho = 2^{1/2} l \operatorname{tg} x, \quad d\rho = 2^{1/2} l \cos^{-2} x\, dx, \quad x = \operatorname{arcctg}(\rho 2^{-1/2}/l)$$

and integrating the resulting integral, we have finally

$$\varphi = kF(\arcsin\{\rho[\rho^2 + 2l^2(1-k^2)]^{-1/2}\}, k) + \varphi_0 \tag{28}$$

Formulas (27) and (28) solve in a general form the given problem. The second determines the connection between ρ and φ, i.e., the equation of the trajectory. Formula (27) defines, in nonexplicit form, the distance ρ of the moving point from the center as a function of time. Notice that the angle φ is always changed over time in a monotonic way—from (16) it is seen that $\dot{\varphi}$ does not change sign.

5. THE TYPE OF PARTICLE MOTION IN QUANTUM SPACE

As in Newtonian mechanics, the particle trajectory given by (28) takes different forms depending on the initial conditions (19) and (21), i.e., on the parameter k in (26). We distinguish several possibilities of interest. From (28) it is easily seen that the connection between the quantities φ and ρ has a definite physical meaning if $\rho[\rho^2 + 2l^2(1-k^2)]^{-1/2} \le 1$. This inequality imposes on the parameter k the restriction $k \le 1$. Thus, the physical conditions of the problem give $0 \le k \le 1$.

1. First we consider the case when $k = 0$. Before discussion of this limit, we make some comments concerning the value ω_0. We assume that ω_0 depends on l, and there may be some link between them. In other words, the condition $l = 0$ gives rise to $\omega_0 = 0$. In contrast, if $\omega_0 \neq 0$ even at $l = 0$, i.e., for the usual classical case, then from (18) and the initial condition $\rho(t)_{t=0} = \rho_0$ it follows that

$$\dot{\varphi} = \frac{(\rho_0^2 + 2l^2)\omega_0}{\rho^2 + 2l^2}\bigg|_{l=0} = \left(\frac{\rho_0}{\rho}\right)^2 \omega_0$$

and this in turn involves a complication of the particle's trajectory analogous to what was obtained above.

Such a situation is completely ruled out by classical mechanical principles. We present here a simple connection between ω_0 and l:

$$\omega_0 = l v_0/\lambda^2 \qquad \text{or} \qquad \omega_0 = l^2 v_0/\lambda^3 \tag{29}$$

where λ is some typical length, which may be identified with the dimension of an atom, $\lambda = a \approx 10^{-8}$ cm, and the Planck length, $\lambda = l_{Pl} = (hG/c^3)^{1/2} = 10^{-33}$ cm, in classical and quantum physics, respectively. In the last case the parameter l is determined by the unit of the Planck length, $l = n l_{Pl}$, $n = 1, 2, 3, \ldots$.

Thus, by definition (26) and assumption (29), the equality $k = 0$ is achieved at $l = 0$ even for $\rho_0 \neq 0$. Notice that $a = \rho_0/l$ in (26) in accordance with initial condition (19). In the case $k = 0$, from (28) it immediately follows that $\varphi = \text{const} = \varphi_0$. At the same time, equation (27) takes the form

$$\rho(t) = \rho_0 + v_0 t \tag{30}$$

since $E(\varphi, 0) = F(\varphi, 0) = \varphi$ and

$$\frac{2^{1/2} l}{[v_0^2 + (a^2+2)\omega_0^2 l^2]^{1/2}} \frac{\rho(t)}{2^{1/2} l}\bigg|_{l\to 0} = \frac{\rho(t)}{v_0}$$

$$C_3|_{l\to 0} = \lim_{l\to 0} \frac{2^{1/2} l}{[v_0^2 + (a^2+2)\omega_0^2 l^2]^{1/2}} \left[\frac{\rho_0}{2^{1/2} l}\left(1 - \frac{2k^2 l^2}{\rho_0^2 + 2l^2}\right)^{1/2} + E\left(\operatorname{arcctg}\frac{\rho_0 2^{-1/2}}{l}, k\right) - F\left(\operatorname{arcctg}\frac{\rho_0 2^{-1/2}}{l}, k\right)\right]$$

$$= \frac{\rho_0}{v_0}$$

So, we see that the case $k = 0$ is just the classical situation where a particle moves along a rectilinear trajectory given by a ray $\varphi = \varphi_0$ (Figure 1a).

2. For the case $k \ll 1$ in order to expose the general pattern of particle trajectory one can use approximate integration of the motion equation (17). Instead of (27) and (28), we have the following approximate equations:

$$t = \frac{2^{1/2} k}{\omega_0 l (2 + a^2)} \left(\rho - \rho_0 + l k^2 2^{-1/2} \operatorname{arcctg} \frac{2^{1/2} l (\rho - \rho_0)}{2l^2 + \rho\rho_0}\right) \tag{31}$$

and

$$\varphi = k \operatorname{arcctg}(\rho 2^{-1/2}/l) + \varphi_0$$

or

$$\rho(\varphi) = 2^{1/2} l \operatorname{tg}[(\varphi - \varphi_0)/k] \tag{32}$$

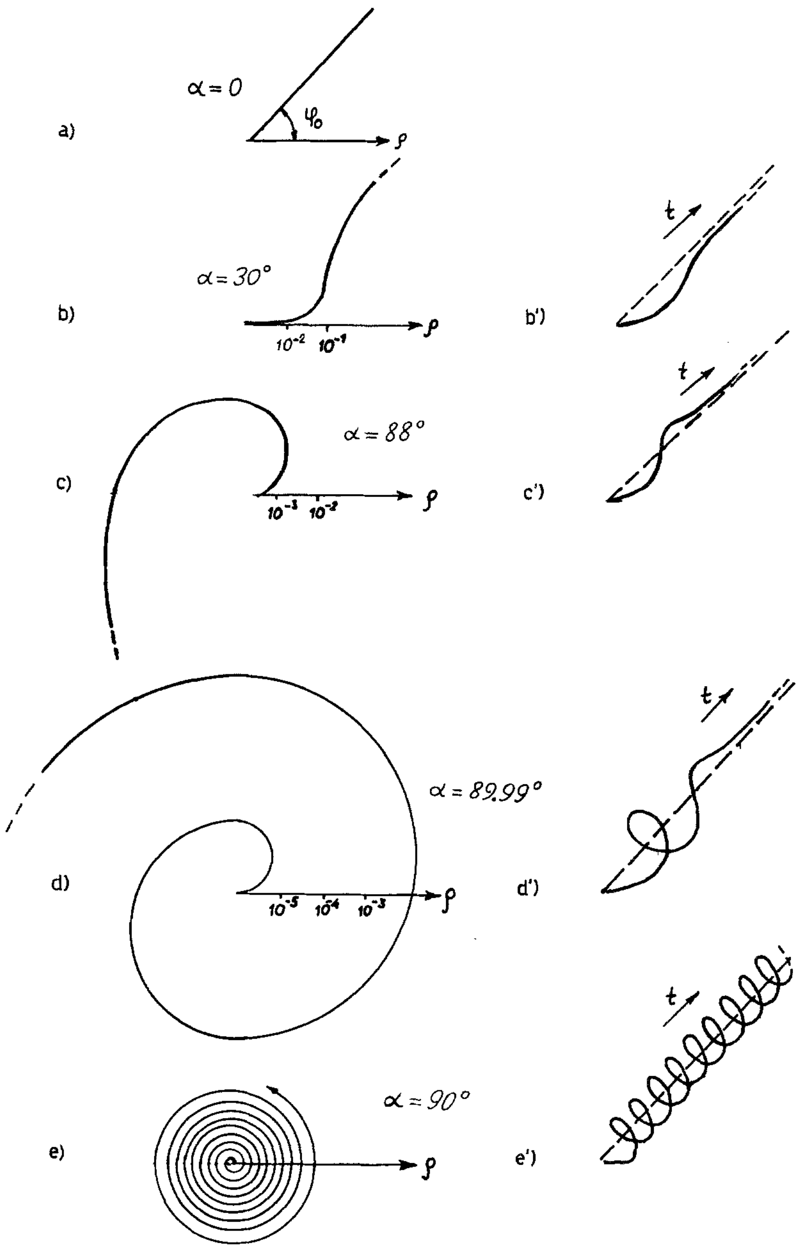

Fig. 1. Different types of particle motion, depending on initial conditions in quantum space.

Thus, we see that in the limit $k \ll 1$ the type of particle motion differs slightly from rectilinear, as in the classical mechanical case.

3. When value of k is increased, the deviation of the particle trajectory from rectilinear becomes more appreciable and it begins to whirl. For example, Figures 1b–1d show the particle trajectory for $\alpha = 30$, 88, and 89.99°, where $k = \sin\alpha$. We see that for $\alpha = 30$, 88, and 89.99 the maximum value of the angle φ is 48, 271, and $\sim 540°$, respectively, which in turn correspond to one-quarter, three-quarters, and more than one full turn, approximately.

4. From the physical point of view, a very interesting case is the limit $k = 1$ or $\alpha = 90°$. In this limiting case, the number of twisted loops (or orbits) becomes infinite (Figure 1e) and the particle is subject to rotational motion at all times. Moreover, the type of rotational motion does not depend on the value of ρ, i.e., for any distance ρ from center, the particle moves along a spiral-like trajectory. However, spiral-like behavior of the particle takes place in the domain characterized by the parameter l of the theory. In other words, the amplitude l of this twisted trajectory determines the maximum deviation from a rectilinear trajectory.

Finally, to present the general pattern of the particle motion over time, we illustrate on the right-hand side of Figure 1 the possible types of particle trajectory corresponding to the left side of Figure 1.

6. DISCUSSION OF THE RESULTS

Thus, as shown above, in quantum space the Lagrangian function or energy of the particle is determined by two terms $E = T + R_l$, where

$$T = m\mathrm{v}^2/2, \qquad R_l = l^2M^2/r^4 + (l^2M_z^2/\rho^4)(z/r)^2$$

In spherical coordinates these take the form

$$T = mv^2/2 = (m/2)(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2)$$
$$R_l = ml^2(\dot{\theta}^2 + \dot{\varphi}^2)$$

or in the two-dimensional case (in polar coordinates)

$$T = (m/2)(\dot{\rho}^2 + \rho^2\dot{\varphi}^2), \qquad R_l^{(2)} = ml^2\dot{\varphi}^2$$

We call R_l and $R_l^{(2)}$ the torsion torque of the particle, which appear due to the quantum nature of space at small distances. In the presence of torsion torque the particle's trajectory is twisted and the particle moves along spiral-like lines.

Our hope is that if the quantum nature of space does indeed exist at small distances, then its discovery may be made by the study of a particle's trajectory. In terms of a suitable choice of initial conditions, one can obtain the case of $k=1$ and at which the particle undergoes rotational motion at all times. It should be noted that it is quite possible to observe the physical effects caused by this twisting motion of the particle, especially for the motion of a charged relativistic particle in an external electromagnetic field. We suggest that the influence of the quantum structure of space-time on the particle behavior is crucial in the relativistic case. This problem requires separate investigation and is the subject of our future work.

Due to the quantum structure of space the particle trajectory deviates from rectilinear at the classical level; one can introduce the fundamental assumption that in quantum space the microparticle's position is not definite and the particle cannot hit a definite place in space (for the generalized complex case different from that considered above). It only occupies, at least, some domain characterized by the parameter λ or l. We now find the amplitude of this deviation from the point the particle would arrive at exactly if space possessed a nonquantum character. Let the particle move with constant velocity v_0 along the z axis. If space is nonquantum, then after the time $t_0=z_0/v_0$ the particle hits the target (slot) at the point $z=z_0$ exactly (Figure 2a) Here there is no deviation along the x and y axes. However, in accordance with the assumption that space possesses quantum structure, the particle deviates from the initial position and traces a circle of radius ρ determined by equation (27), where one should put $t=t_0$ and $v_0=0$, since we originally suggested that there was no motion along the x and y axes. In this sense, the deviation is a pure quantum space effect (see Figure 2b).

The amplitude of the deviation in which we are interested is given by (27), in which we put $t=t_0$, $v_0=0$, and $\rho \gg 1$, since in the classical mechanical case, the influence due to quantum space may be observable if the deviation is much larger than the value of l.

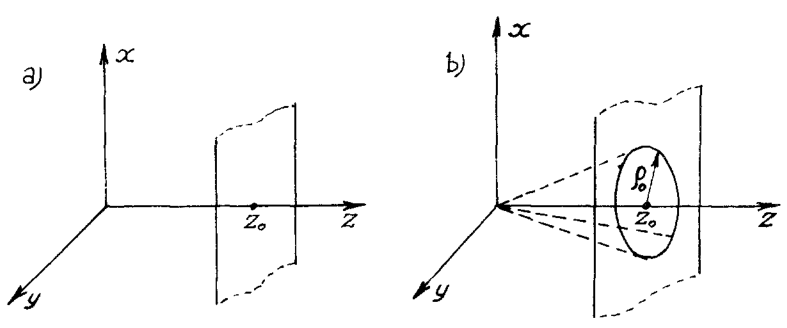

Fig. 2. Illustration of particle position according to the assumption of particle motion in (a) nonquantum or (b) quantum space.

It should be noted that from high-energy experiments it follows that $l \lesssim 10^{-16}$ cm (see, for example, Namsrai, 1985a,b). Thus, assuming $\rho \gg l$ and $\rho_0 = al = 0$ in (27), we have

$$t = \frac{2^{1/2} l}{2^{1/2} \omega_0 l} \left[\frac{\rho 2^{-1/2}}{l} + E(0, 2^{-1/2}) - F(0, 2^{-1/2}) \right] \approx \frac{\rho 2^{-1/2}}{\omega_0 l}$$

and

$$\rho(t_0) = 2^{1/2} \omega_0 t_0 l$$

Let an idealized classical object, a small bullet with initial velocity $v_0 = 1000\ \mathrm{m/sec} = 10^5\ \mathrm{cm/sec}$, move along the z axis and hit the target after $t_0 = 100$ sec. Now, the following question arises; How far does its rectilinear trajectory deviate after $t_0 = 100$ sec? According to (29), we get

$$\omega_0 = l v_0 / \lambda^2 = 10^5/\mathrm{sec} = 0.1\ \mathrm{MHz}$$

and therefore

$$\rho(t_0) = 14 \times 10^{-9}\ \mathrm{cm}$$

Thus, this value is completely negligible from the classical experimental point of view.

In conclusion, we note that in the microworld where physical processes take place at small distances an effect analogous to that discussed above should play an important role and due to torsion torque the trajectory of microparticles should become very tortuous. Moreover, it is quite possible that the essence of an observable quantum process may be understood as Brownian-type stochastic motion taking place in quantum space-time at small distances. Thus, this may open the door for a stochastic foundation of quantum mechanics [see, for example, Prugovecki (1984) and Namsrai (1985a)], as initiated by A. Einstein and L. de Broglie, seeking to describe quantum processes by means of subquantum deterministic motions.

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